ASSESSMENT OF EFFECT OF HEAT OF INTERNAL FRICTION ON CHARACTERISTICS OF STRUCTURED FLOW OF VISCOPLASTIC LIQUID IN A ROUND TUBE

V. E. Pervushin

UDC 532.5:532.135

A structured pressure flow of viscoplastic liquid in an infinite round tube is examined. Energy dissipation is given due consideration; it is assumed that the temperature dependences of the plastic viscosity and the ultimate shear stress are exponential. The solution obtained when the parameter characterizing the rate of production of heat of internal friction is assumed to be small indicates that energy dissipation has a sgnificant effect on the local flow characteristics and the hydraulic-resistance coefficient, for which an expression suitable for engineering calculations is given.

There have been investigations [1-7] of steady structured pressure flows of Newtonian and non-Newtonian liquids in which the energy dissipation and the variation of the rheological characteristics with temperature have been taken into account.

Kaganov [3] showed that there are critical pressure gradients above which a steady flow of Newtonian liquid is impossible. Some authors [4-6] have shown that a hydrodynamic thermal explosion can occur in flows of Newtonian [4] and non-Newtonian [5, 6] liquids.

It is of interest to solve this problem in the region of parameters in which the flow and heat-transfer regimes are steady. If a large error is to be avoided we must take temperature dependences of the rheological characteristics of the liquid which are similar to the experimental relationships.

In the investigation of a pressure flow of viscoplastic liquid in [6, 7], where energy dissipation received due consideration, it was assumed that the plastic viscosity varies with temperature in accordance with a hyperbolic law, and the ultimate shear stress varies with temperature in accordance with the same law [6], or is constant [7].

Such viscoplastic liquids as paraffin oils are characterized by a strong temperature dependence of their rheological characteristics [8, 9]; this dependence is satisfactorily approximated by an exponential curve [9, 10].

A pressure flow of viscoplastic liquid for this case of variation of the rheological characteristics with temperature has not been investigated.

We consider a steady structured flow of viscoplastic liquid with a Shvedov–Bingham rheological equation due to a pressure difference ($-dp/dz$) in a round tube of radius R along the z axis. A constant temperature T_0 is maintained at the wall, the temperature gradient along the flow is zero, and the nonisothermicity of the flow is due to energy dissipation.

We assume that the variation of the plastic viscosity and ultimate shear stress can be approximated by an exponential function

$$\eta(T) = \eta_0 \exp[-\beta_1 (T - T_0)], \quad \tau_0(T) = \tau_0 \exp[-\beta_2 (T - T_0)] \tag{1}$$

Moscow. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 3, pp. 79-85, May-June, 1974. Original article submitted May 21, 1973.

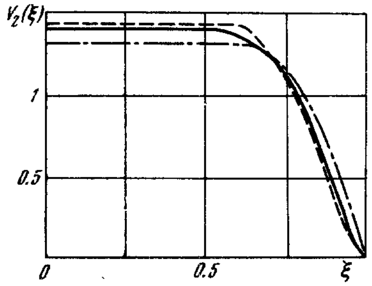

Fig. 1

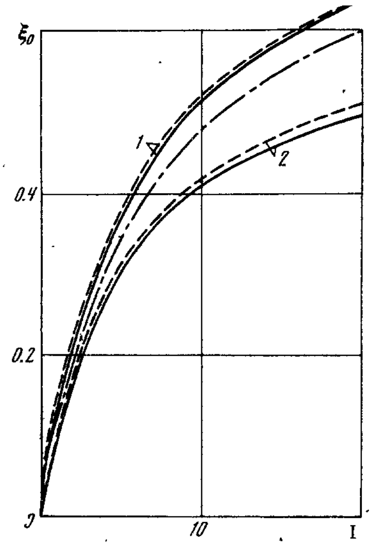

Fig. 2

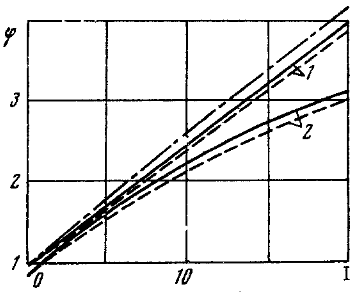

Fig. 3

where η_0 and τ_0 are the plastic viscosity and ultimate shear stress, calculated at the wall temperature; β_1 and β_2 are constants.

The system of equations of motion and heat conduction in dimensionless variables in the case of (1) has the form

$$V_1 \equiv \text{const}, \quad \frac{1}{\xi}\frac{d}{d\xi}\left(\xi\frac{d\theta_1}{d\xi}\right) = 0, \quad 0 \leqslant \xi \leqslant \xi_0 \tag{2}$$

$$\frac{1}{\xi}\frac{d}{d\xi}\left[\xi e^{-\theta_2}\left(-\mathrm{I}\, e^{-(\beta-1)\theta_2} + 2\frac{dV_2}{d\xi}\right)\right] - \frac{\mathrm{Re}}{2}\frac{dP}{d\zeta} = 0, \quad \xi_0 \leqslant \xi \leqslant 1 \tag{3}$$

$$\frac{1}{\xi}\frac{d}{d\xi}\left(\xi\frac{d\theta_2}{d\xi}\right) + \alpha e^{-\theta_2}\left(-\mathrm{I}\, e^{-(\beta-1)\theta_2} + 2\frac{dV_2}{d\xi}\right)\left(\frac{dV_2}{d\xi}\right) = 0, \quad \xi_0 \leqslant \xi \leqslant 1 \tag{4}$$

$$\mathrm{I}\, e^{-\beta\theta_1} \geqslant -\frac{\mathrm{Re}}{4}\frac{dP}{d\zeta}\xi, \quad 0 \leqslant \xi \leqslant \xi_0 \tag{5}$$

$$\xi = r/R, \quad \xi_0 = r_0/R, \quad \zeta = z/R, \quad V_i = v_{iz}/\langle v\rangle, \quad P = p/(\rho\langle v\rangle^2/2), \quad \beta = \beta_2/\beta_1$$
$$\theta_i = \beta_1(T_i - T_0), \quad \mathrm{Re} = 2R\langle v\rangle\rho/\eta_0, \quad \mathrm{I} = 2\tau_0 R/\langle v\rangle\eta_0$$
$$\alpha = \beta_1\langle v\rangle^2\eta_0/2kJ$$

Here i = 1 and 2 for the core and viscoplastic region, respectively, r is the variable radius, r_0 is the core radius, v_{iz} is the longitudinal velocity, $\langle v \rangle$ is the mean velocity, determined by the ratio of the volume flow per second to the cross-sectional area of the tube, p is the pressure, ρ is the density of the liquid, Re is the Reynolds number, I is the Ilyushin parameter, α is a parameter characterizing the rate of heat production due to viscous friction, k is the thermal conductivity of the liquid, and J is the mechanical equivalent of heat.

The boundary conditions are

$$d\theta_1/d\xi = 0, \quad \xi = 0 \tag{6}$$

$$V_1 = V_2, \quad \frac{dV_2}{d\xi} = 0, \quad \theta_1 = \theta_2, \quad \frac{d\theta_1}{d\xi} = \frac{d\theta_2}{d\xi}, \quad \xi = \xi_0 \tag{7}$$

$$V_2 = 0, \quad \theta_2 = 0, \quad \xi = 1 \tag{8}$$

To the boundary conditions (6)-(8) we have to add the expression for the dimensionless volume flow rate

$$V_1\xi_0^2 + 2\int_{\xi_0}^{1} \xi V_2(\xi)\, d\xi = 1 \tag{9}$$

The solution of (2), taken in conjunction with (6), is

$$\theta_1 = A \tag{10}$$

where A is a constant.

For the core radius we obtain from (5) and (10)

$$\xi_0 = 4\mathrm{I}\, \exp(-\beta A)\,\mathrm{Re}^{-1}(-dP/d\zeta)^{-1} \tag{11}$$

Integration of (3) in conjunction with the second boundary condition (7) and the expression for the core (11) gives

$$dV_2/d\xi = {}^1/_2\left(-\varkappa\xi e^{\theta_2} + \mathrm{I}\, e^{-(\beta-1)\theta_2}\right) \tag{12}$$

$$\varkappa = -{}^1/_4\,\mathrm{Re}\, dP/d\zeta \tag{13}$$

We will henceforth assume that the mean flow rate, or the parameters I, α, and Re, are known.

Substitution of (12) in (4) leads to the equation

$$\frac{d^2\theta_2}{d\xi^2} + \frac{1}{\xi}\frac{d\theta_2}{d\xi} + \frac{\alpha}{2}\left(\varkappa^2\xi^2 e^{\theta_2} - \mathrm{I}\varkappa\xi e^{-(\beta-1)\theta_2}\right) = 0 \tag{14}$$

The solution of (14) is not expressed in quadratures, but can be obtained numerically. In (14) there are three independent parameters: I, α, and β. [The parameter $\varkappa$, introduced above, is expressed as a function of I, α, and β by means of (9).] This greatly complicates the tabular or graphic representation of the numerical solution.

In many cases of practical interest we can assume that $\alpha \ll 1$.

We construct an approximate solution of the problem by the perturbation method [11], putting

$$\theta_i = \theta_{i,0} + \alpha\theta_{i,1} + \alpha^2\theta_{i,2} + \dots, \quad V_i = V_{i,0} + \alpha V_{i,1} + \alpha^2 V_{i,2} + \dots \quad \xi_0 = \xi_{0,0} + \alpha\xi_{0,1} + \alpha^2\xi_{0,2} + \dots,$$
$$\varkappa = \varkappa_0 + \alpha\varkappa_1 + \alpha^2\varkappa_2 + \dots \tag{15}$$

By substituting expansions (15) in (12), (14), and the boundary conditions, and equating coefficients of equal powers of α, we obtain a recurrent system of linear boundary-value problems for the zero, first, and subsequent approximations. For $\alpha = 0$ the temperature distribution is uniform: $\theta_{i,0} = 0$. The zero approximation $V_{i,0}$ is the known solution [12]

$$V_{1,0} = \frac{1}{2}\, I(\xi_{0,0} - 1) - \frac{\varkappa_0}{4}(1 - \xi_{0,0}^2), \quad V_{2,0} = \frac{1}{2}\, I(\xi - 1) + \frac{\varkappa_0}{4}(1 - \xi^2), \quad \varkappa_0 = I/\xi_{0,0}. \tag{16}$$

To determine the core radius in the isothermal case ($\alpha = 0$) we use an equation [12], which in the adopted symbols have the form

$$\xi_{00}^4 - 3\chi\xi_{0,0} + 3 = 0, \quad \chi = 4/3 + 8/I \tag{17}$$

Equation (17) has a single positive root less than unity [12].

The first approximation is found from the solution of the following linear boundary-value problem:

$$\frac{dV_{2,1}}{d\xi} = \frac{1}{2}[-\varkappa_0\xi - (\beta - 1)\, I]\,\theta_{2,1} - \frac{1}{2}\xi\varkappa_1 \tag{18}$$
$$\frac{d^2\theta_{2,1}}{d\xi^2} + \frac{1}{\xi}\frac{d\theta_{2,1}}{d\xi} + \frac{1}{2}(\varkappa_0^2\xi^2 - \varkappa_0\xi) = 0 \tag{19}$$
$$V_{1,1} = V_{2,1}, \quad \theta_{1,1} = \theta_{2,1}, \quad d\theta_{2,1}/d\xi = 0; \quad \xi = \xi_{0,0} \tag{20}$$
$$V_{2,1} = 0, \quad \theta_{2,1} = 0, \quad \xi = 1 \tag{21}$$
$$\varkappa_0\xi_{0,1} + \varkappa_1\xi_{0,0} = -\beta\, I\, \theta_{1,1} \tag{22}$$
$$V_{1,1}\xi_{0,0}^2 + 2\int_{\xi_{0,0}}^{1} \xi V_{2,1}(\xi)\, d\xi = 0 \tag{23}$$

Integrating (19), using the third boundary condition (20) and the condition (21), we obtain

$$\theta_{2,1} = \frac{I^2}{2\xi_{0,0}^2}\left(\frac{1-\xi^4}{16} - \frac{(1-\xi^3)\,\xi_{0,0}}{9} - \frac{\xi_{0,0}^4 \ln\xi}{12}\right) \tag{24}$$

The function $\theta_{1,1}$ is found from the second boundary condition (20). Substituting (24) in (18) and integrating in conjunction with the first boundary condition (21), we obtain

$$V_{2,1} = \frac{1}{2}\left\{\frac{\varkappa_1}{2}(1-\xi^2) + \varkappa_0\int_{\xi}^{1}\xi\theta_{2,1}d\xi + I(\beta-1)\int_{\xi}^{1}\theta_{2,1}d\xi\right\} \tag{25}$$

The integrals in (25) can be calculated, but in view of the unwieldy final expressions it is more convenient to obtain the result for a specific value of I.

From (25) and the first boundary condition (20) we find $V_{1,1}$. Substituting (25) in (23) and integrating by parts we find

$$\varkappa_1 = -4/(1-\xi_{0,0}^4)\left[\varkappa_0\int_{\xi_{0,0}}^{1}\xi^3\theta_{2,1}d\xi + I(\beta-1)\int_{\xi_{0,0}}^{1}\xi^2\theta_{2,1}d\xi\right] \tag{26}$$

The value of $\xi_{0,1}$ is determined from (22).

For the second approximation we have the boundary-value problem

$$dV_{2,2}/d\xi = -{}^1/_2[\varkappa_0\xi + I(\beta-1)]\,\theta_{2,2} + {}^1/_4[-\varkappa_0\xi + (\beta-1)^2\, I]\,\theta_{2,1}^2 - {}^1/_2\xi\varkappa_1\theta_{2,1} - {}^1/_2\varkappa_2\xi \tag{27}$$
$$\frac{d^2\theta_{2,2}}{d\xi^2} + \frac{1}{\xi}\frac{d\theta_{2,2}}{d\xi} + \frac{1}{2}[\varkappa_0^2\xi^2 + (\beta-1)\, I\varkappa_0\xi]\,\theta_{2,1} + {}^1/_2(2\varkappa_0\xi^2 - I\xi)\,\varkappa_1 = 0 \tag{28}$$
$$\theta_{1,2} = \theta_{2,2}, \quad d\theta_{2,2}/d\xi = 0, \quad \xi = \xi_{0,0} \tag{29}$$
$$V_{2,2} = 0, \quad \theta_{2,2} = 0, \quad \xi = 1 \tag{30}$$
$$\varkappa_0\xi_{0,2} + \varkappa_1\xi_{0,1} + \varkappa_2\xi_{0,0} = -I\beta\,(\theta_{1,2} - {}^1/_2\beta\theta_{1,1}^2) \tag{31}$$
$$V_{1,2}\xi_{0,0}^2 + 2\int_{\xi_{0,0}}^{1}\xi V_{2,2}d\xi = 0 \tag{32}$$

$$V_{1,2} = V_{2,2}(\xi_{0,0}) + \xi_{0,1} \frac{dV_{2,1}}{d\xi}(\xi_{0,0}) + \frac{\xi_{0,1}^2}{2} \frac{d^2V_{2,0}}{d\xi^2}(\xi_{0,0}) \tag{33}$$

We give the result

$$\theta_{2,2} = -{}^1/_2 \ln\xi \int\limits_{\xi_{0,0}}^{\xi} f_1\theta_{2,1} d\xi - {}^1/_2 \int\limits_{\xi}^{1} \ln\xi f_1\theta_{2,1} d\xi + \varkappa_1/2\,[\mathrm{I}\ \xi_{0,0}^3/6 \ln\xi + \varkappa_0(1-\xi^4)/8 - \mathrm{I}\,(1-\xi^3)/9] \tag{34}$$

$$V_{2,2} = {}^1/_4 \varkappa_2 (1-\xi^2) + {}^1/_2 \int\limits_{\xi}^{1} (f_1\theta_{2,2}/\varkappa_0\xi^2 - f_2\theta_{2,1}^2/2 + \varkappa_1\xi\theta_{2,1})\, d\xi \tag{35}$$

$$\varkappa_2 = -4/(1-\xi_{0,0}^4)\Big[\int\limits_{\xi_{0,0}}^{1} \xi^2 (f_1\theta_{2,2}/\varkappa_0\xi^2 - f_2\theta_{2,1}^2/2 + \varkappa_1\xi\theta_{2,1})\, d\xi + \xi_{0,0}^2\xi_{0,1}^2\varkappa_0/2\Big]$$

$$f_1 = \varkappa_0{}^2\xi^3 + \mathrm{I}\ (\beta-1)\,\varkappa_0\xi^2, \quad f_2 = -\varkappa_0\xi + (\beta-1)^2\mathrm{I} \tag{36}$$

Equations (16), (17), (22), and (24)-(26) give the solution of the problem in the first approximation, and (31), (33), and (34)-(36) give it in the second approximation.

Despite the rather unwieldy form, the solution for each specific case can be obtained relatively easily, particularly by computer, since the procedure reduces to finding the root of Eq. (17) and calculating definite integrals.

We should mention one special feature of the above-described linearization of system (1)-(9) due to the behavior of the zero approximation and its derivatives at point ξ_{00}. The core radius corresponding to the k-th approximation is that of the (k−1)-th approximation.

Computer calculations in the range of parameters ($0 < \mathrm{I} \leq 40$, $0 \leq \beta \leq 1$), showed that energy dissipation can significantly alter the local and integral flow characteristics. The differences in the solutions of the first and second approximations are very insignificant.

Some of the results of the calculations are illustrated in Figs. 1-3. On all the figures the isothermic solutions are represented by dot-dash curves, the first-approximation solutions by dashed curves, and the second-approximation solutions by continuous curves.

Figure 1 shows characteristic velocity profiles calculated for the following values of parameters: $\mathrm{I} = 20$, $\alpha = 0.1$, $\beta = 1$; $\alpha = 0$ corresponds to the isothermic Buckingham profile (16). The presented curves clearly show the effect of internal heating of the liquid, which leads to an increase in the relative flow velocity in the core region and its reduction near the wall. The greatest difference in the velocity profiles of the first and second approximations is found in a narrow region contiguous with the core boundary, which can be attributed to the above-mentioned feature of the linearization of the initial system.

Figure 2 shows a graph of the variation of the dimensionless core boundary in (ξ_0, I) coordinates in the form of a family of curves with parameter β, calculated for the single value $\alpha = 0.1$ (the values of β corresponding to curves 1 and 2 are 0 and 1). The relationship between the core radius and the Ilyushin parameter, calculated from (17), corresponds to $\alpha = 0$. As Fig. 2 shows, the core radius can be greater or smaller than its value in isothermic flow (for a fixed mean velocity), depending on the value of β. This is due to the fact that in the case where the ultimate shear stress is independent of the temperature ($\beta = 0$) the reduction of the liquid viscosity due to dissipation leads to reduction of the shear stress and enlargement of the core. As β increases, reduction of the ultimate shear stress becomes the dominant effect.

For the hydraulic-resistance coefficient of a round tube, using (13), we obtain

$$\lambda = 2R\,(-dp/dz)/\rho\,\langle v^2\rangle/2, \quad \lambda = 64/\mathrm{Re}\ (\varkappa_0/8 + \alpha\varkappa_1/8 + \alpha^2\varkappa_2/8) = 64/\mathrm{Re}\ \varphi(\alpha,\ \mathrm{I},\ \beta) \tag{37}$$

The first term of the sum in (37) is the hydraulic-resistance coefficient for isothermal flow [13] of a viscoplastic liquid in a round tube; the second and third terms of this sum are corrections to the first and second approximations, respectively, for dissipative heating of the liquid.

Function $\varphi(\alpha, \mathrm{I}, \beta)$ is shown in Fig. 3 in the form of a family of curves with parameter β. (The correspondence between the number of the curve and β is the same as in Fig. 2). All the curves were calculated for one value of α(0.1). The dot-dash curve represents function φ_1(I), which is a solution of Eq. (17), where the correspondence $\xi_{0,0} = \mathrm{I}/\varkappa_0$ must be taken into account.

As Fig. 3 shows, the hydraulic-resistance coefficient for a flow with dissipative heating is lower than in the case of isothermic flow or a flow of liquid with constant rheological characteristics. The differences can become very large with increase in β, even when the dissipative parameter has a very low value. This circumstance must be taken into account in accurate measurements on capillary viscosimeters.

The experimental results are usually represented in the form of a relationship between the mean shear velocity gradient $\langle dv_z/dr \rangle = 4\langle v \rangle /R$ and the wall shear stress $\tau_W = {}^1/_2 R(-dp/dz)$. In the case of a viscoplastic liquid this relationship tends asymptotically to a straight line [14]. If experiment shows a deviation from a straight line, we can infer that the Shvedov–Bingham model does not represent the rheological behavior of the liquid. In the case of liquids whose rheological characteristics are very sensitive to temperature change, however, the deviation can be due to the dissipative effect.

The results obtained above allow this effect to be taken into account. For a given mean flow velocity and known rheological constants τ_0 and η_0 the pressure drop, with allowance for energy dissipation, is given by formula (37).

I thank V. I. Maron for supervision of the work.

LITERATURE CITED

1. H. Hausenblas, "Die nichtisoterme laminare Strömung einer zähen Flüssigkeit durch enge Spalte und Kapillarröhren," Ingr.-Arch., 18, No. 3 (1950).
2. S. A. Regirer, "Some thermohydrodynamic problems of a steady one-dimensional flow of viscous drop-forming liquid," Prikl. Matem. Mekh. 21, No. 3 (1957).
3. S. A. Kaganov, "Steady laminar flow of incompressible liquid in a flat channel and around cylindrical tube with due regard to friction heat and variation of viscosity with temperature," Zh. Prikl. Mekhan. Tekh. Fiz., No. 3 (1962).
4. S. A. Bostandzhiyan, A. G. Merzhanov, and S. I. Khudyaev, "The hydrodynamic thermal explosion," Dokl. Akad. Nauk SSSR, 163, No. 1 (1965).
5. S. A. Bostandzhiyan and S. M. Chernyaeva, "The hydrodynamic thermal explosion of a non-Newtonian liquid," Dokl. Akad. Nauk SSSR, 170, No. 2 (1966).
6. V. P. Belomyttsev and N. N. Gvozdkov, "Loss of thermal stability of motion of a viscoplastic material," Dokl. Akad. Nauk SSSR, 170, No. 2 (1966).
7. Ya. M. Rasizade and R. M. Mamedov, "Effect of heat of internal friction on hydraulic characteristics of a structured flow of viscoplastic media in tubes," Izv. Vysshikh Uchebn. Zavedenii, Neft i Gaz, No. 12 (1968).
8. E. G. Barry, "Pumping non-Newtonian waxy crude oils," J. Inst. Petrol., 57, No. 554 (1971).
9. F. F. Abuzova, V. F. Novoselov, and P. I. Tungunov, "Selection of equation for temperature dependence of static shear stress of paraffin oils," Izv. Vysshikh Uchebn. Zavedenii, Neft i Gaz, No. 1 (1972).
10. P. A. Filonov, Movement of Oil through Pipes [in Russian], Neftyanoe Izd-vo, Moscow–Leningrad (1930).
11. G. N. Polozhii, N. A. Pakhareva, I. Z. Stepanenko, P. S. Bondarenko, and I. M. Velikoivanenko, Mathematical Practicum [in Russian], Fizmatgiz, Moscow (1960).
12. A. Kh. Mirzadzhanzade, Questions of Hydrodynamics of Viscoplastic and Viscous Liquids in Application to Oil Extraction [in Russian], Azerneftneshr, Baku (1959).
13. B. I. Mitel'man and G. D. Rozenberg, "Structured flow of viscoplastic liquid through a cylindrical tube of round section," Izv. Akad. Nauk SSSR, Otd. Tekhn. Nauk, Mekhan. i Mashinostr., No. 4 (1961).
14. M. Reiner, Rheology, Springer-Verlag, Berlin (1958).